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Time-like geodesics in the first Tomimatsu–Sato metric

Joaquin Diaz Alonso†

Groupe d'Astrophysique Relativiste, Observatoire de Meudon, Meudon, France

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Abstract. A numerical study of the time-like geodesics in the first axially symmetric solution of the Einstein vacuum field equations given by Tomimatsu and Sato is made. The study is made on the equatorial plane, following the method employed by Bardeen, Press and Teukolsky in the Kerr field.

1. Properties of Tomimatsu–Sato metrics

The stationary axially symmetric solutions of the Einstein vacuum field equations can be written in the Papapetrou form:

$$ds^2 = -f(dt-\omega\, d\phi)^2+\frac{\rho^2}{f}\, d\phi^2+\frac{e^{2\Gamma}}{f}(d\rho^2+dz^2). \tag{1}$$

We obtain the first Tomimatsu–Sato (TS) solution (Tomimatsu and Sato 1972, 1973a) if:

$$f=\frac{A}{B}; \qquad \omega=\frac{2mq}{A}(1-y^2)C;$$
$$e^{2\Gamma}=\frac{A}{P^{\delta^2}(x^2-y^2)^{\delta^2}} \qquad \text{and} \qquad \delta=2, \tag{2}$$

where x, y are prolate spheroidal coordinates, related to canonical coordinates through:

$$\rho=\frac{mP}{\delta}(x^2-1)^{1/2}(1-y^2)^{1/2}$$
$$z=\frac{mP}{\delta}xy, \tag{3}$$

δ is an integer parameter of the TS family ($\delta=1$ Kerr metric, $\delta=2$ first TS metric), m is the gravitational mass, q is a parameter related to the angular momentum by:

† Fellow of Fundacion del Instituto Nacional de Industria, Madrid, Spain.

$J = m^2 q$; and $P = (1-q^2)^{1/2}$. A, B and C are the polynomials:

$$A = P^4(x^2-1)^4 + q^4(1-y^2)^4 - 2P^2q^2(x^2-1)(1-y^2)[2(x^2-1)^2 + 2(1-y^2)^2 + 3(x^2-1)(1-y^2)]$$

$$B = [P^2(x^4-1) - q^2(1-y^4) + 2Px(x^2-1)]^2 + 4q^2y^2[Px(x^2-1) + (Px+1)(1-y^2)]^2 \quad (4)$$

$$C = -P^3x(x^2-1)[2(x^4-1) + (x^2+3)(1-y^2)] - P^2(x^2-1)[4x^2(x^2-1) + (3x^2+1)(1-y^2)] + q^2(Px+1)(1-y^2)^3.$$

The properties of these metrics have been studied elsewhere (Tomimatsu and Sato 1973a, b, Gibbons and Russell-Clark 1973, M A Abramowicz and J P Lasota, unpublished preprint No. 26, Polish Academy of Sciences). We are only interested in the equatorial section ($y = 0$) of the ($\delta = 2$) TS space–time. Here, one finds three critical curves (Tomimatsu and Sato 1973a). When the polynomial A vanishes, the time-like Killing vector $\partial/\partial t$, becomes null, and we enter the ergosphere. This circle is the intersection of the equatorial plane and the exterior infinite redshift surface.

Beyond this circle, there is another circle where both A and B vanish, and where the curvature invariants diverge. This is the ring singularity of the TS metric.

The third peculiar circle, which lies beyond the ring singularity, is the intersection of the false horizon with the equatorial plane, and occurs at $x = 1$.

The axial Killing vector $\partial/\partial\phi$, becomes time-like between the second and the third circles: one therefore finds there closed time-like geodesics and a strong causality violation (Gibbons and Russell-Clark 1973, Abramowicz and Lasota, unpublished).

The physically interesting (or reasonable) region, occurs thus outside the ring singularity (Gibbons and Russell-Clark 1973), and we will study the time-like geodesics in this region only.

The null geodesics have been studied by Tomimatsu and Sato (1973a).

2. Calculation of time-like geodesics

As a consequence of stationarity and axial symmetry, the geodesic equations have two constants of motion, E/μ and Φ/μ; these are the energy and the angular momentum of the test particle which follows the time-like geodesics, divided by its proper mass μ.

The 'energy equation' (Bardeen *et al* 1972, Bardeen 1972) becomes

$$d\left(\frac{d\rho}{d\lambda}\right)^2 = a\left(\frac{E}{\mu}\right)^2 + 2b\frac{E}{\mu} + c \quad (5)$$

with

$$a = P^2B^2(x^2-1) - 16q^2C^2 \qquad b = 8qAC\Phi/\mu m$$

$$c = AB(x^2-1) - 4A^2\Phi^2/\mu^2m^2 \qquad d = AB^2(x^2-1)/P^2x^8$$

where λ is the proper time.

The 'effective potential' is determined by the roots of (5):

$$\frac{E}{\mu} = \frac{-b \pm (b^2 - ac)^{1/2}}{a} = V_{\text{eff}}. \quad (6)$$

The energy of a test particle with respect to a local non-rotating observer (lno) is given by (Bardeen *et al* 1972):

$$E_{\text{lno}} = aE + b. \tag{7}$$

The sign in (6) is determined by the condition that (7) be positive.

The effective potential (6) depends on the angular momentum of the test particle (Φ) and also, on the rotation of the field (P).

Qualitative features of this effective potential are shown in figure 1 for two typical values of the parameter Φ, and the same value of P.

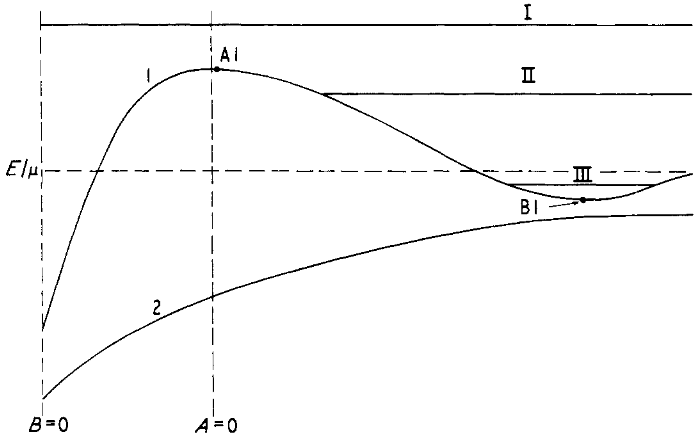

Figure 1. Effective potential for a test particle in a TS gravitational field of given angular momentum (fixed P) and two different values of particle angular momentum (Φ_1 and Φ_2). Straight lines represent different possible trajectories of test particle. Points A1 and B1 correspond to unstable and stable circular orbits.

The straight lines correspond to geodesics of given E and Φ. In the case where the 'effective potential' has a maximum, one has three types of geodesic. If the energy E/μ is higher than the potential barrier, the geodesic goes all the way down to the ring singularity. When E/μ is less than the maximum but more than one, the orbits are quasi-hyperbolic, and when $E/\mu < 1$ (but, of course, exceeds the minimum of the potential), the orbits followed by the free particle are quasi-elliptic.

The energy and position of the maximum and minimum of this potential correspond to unstable and stable circular orbits respectively. The circular orbits, are thus determined by (6) together with the extra condition:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{E}{\mu}\right) = 0. \tag{8}$$

When the potential has neither maxima nor minima, there are obviously no circular orbits, and this is the case for curve 2 of figure 1. For these values of Φ, the 'centrifugal field' is not able to overcome the 'gravitational field', and every geodesic collapses.

In figures 2–5, the behaviour of the stable marginally circular orbits for different values of the parameters is presented. We represent the energy and the angular momentum of these circular orbits as functions of the position for different representative values of the P parameter (or $q = (1-P^2)^{1/2} = a/m$ in Bardeen's notation). Figures 2 and 3

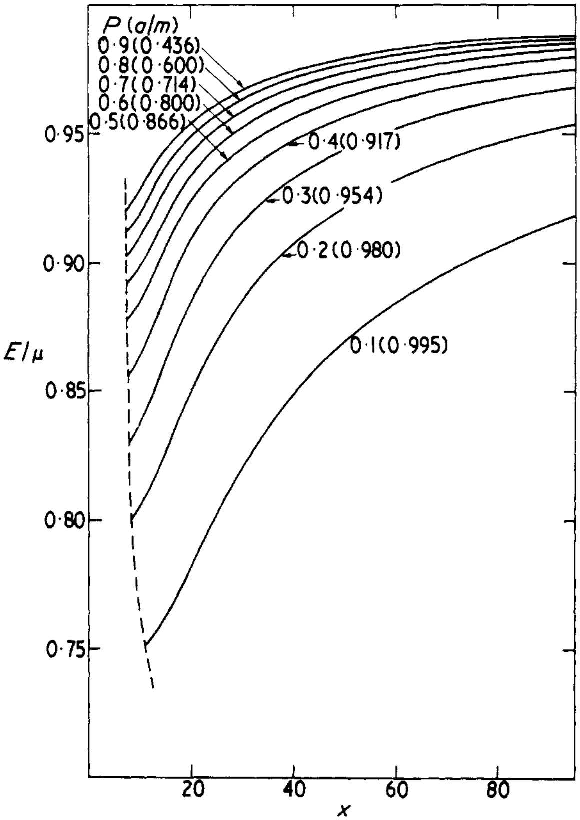

Figure 2. Energy of stable circular orbits as function of particle radial position x, for different representative values of P parameter (or a/m) in the $\Phi > 0$ case.

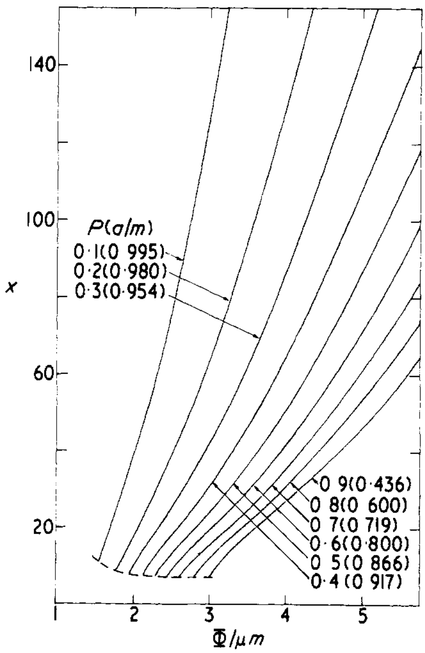

Figure 3. Position of stable circular orbits as function of particle angular momentum Φ for different representative values of field rotation parameter P (or a/m) in the $\Phi > 0$ case.

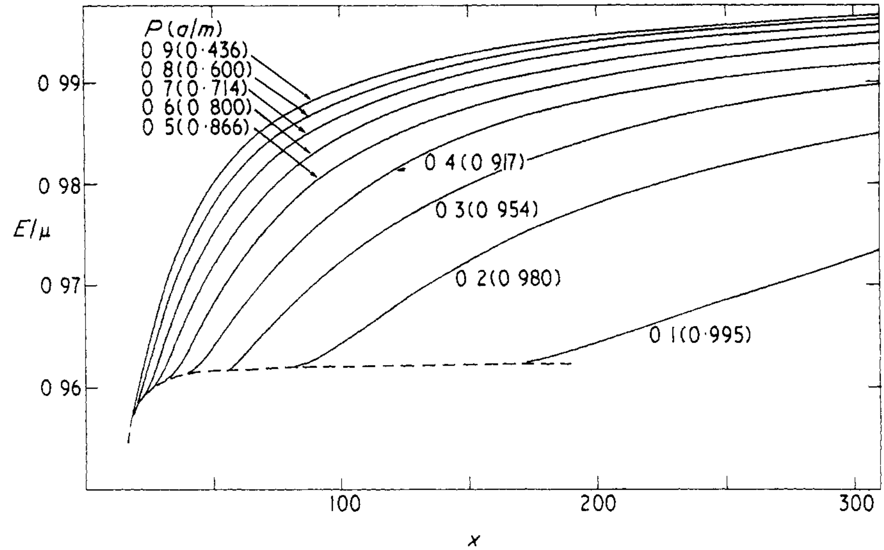

Figure 4. Same quantities as figure 2 in the $\Phi < 0$ case.

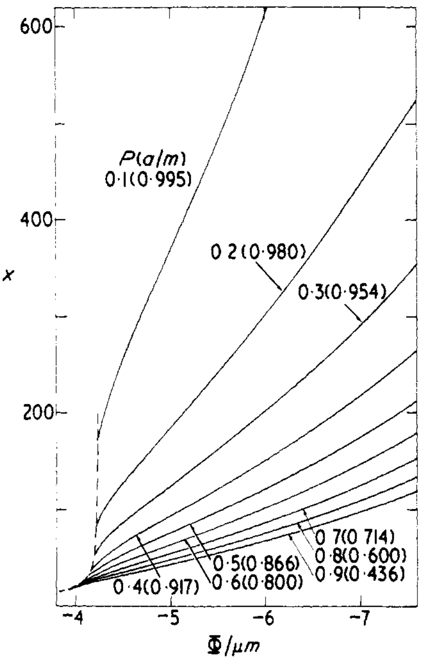

Figure 5. Same quantities as figure 3 in the $\Phi < 0$ case.

show the energy and angular momentum of stable direct circular orbits as functions of position x. Figures 4 and 5 show the same quantities for stable retrograde circular orbits. In these graphs, the broken curves indicate the values of energy, position and angular momentum where circular orbits appear at each value P (or a/m) which measures the field rotation.

In the extreme relativistic case ($P \to 0$ or $a \to m$), the coordinate system (3) becomes singular and it is necessary to make use of the ρ coordinate which represents the physical length. Putting the metrical functions, in the limit $P \to 0$, for a coordinate system (r, θ) defined by:

$$\rho = (r-m)\sin\theta$$

$$z = (r-m)\cos\theta$$

we show that the Kerr and TS metrics coincide, since all the quantities which determine the behaviour of particle trajectories in this limit of TS metrics, are the same as those calculated by Bardeen for the Kerr metric. More precise numerical information, can be obtained from the author.

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